

Scaling limit and convergence of smoothed covariance for gradient models with non-convex potential

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Abstract

A discrete gradient model for interfaces is studied. The interaction potential is a non-convex perturbation of the quadratic gradient potential. Based on a representation for the finite volume Gibbs measure obtained via a renormalization group analysis by Adams, Kotecký and Müller in [AKM] it is proven that the scaling limit is a continuum massless Gaussian free field. From probabilistic point of view, this is a Central Limit Theorem for strongly dependent random fields. Additionally, the convergence of covariances, smoothed on a scale smaller than the system size, is proven.

1 Introduction

We analyze discrete gradient models which are effective models for random interfaces. Let $\Lambda \subset \mathbb{Z}^d$ be a finite set. To each configuration $\varphi \in \mathbb{R}^\Lambda$ ($\varphi(x)$ can be interpreted as the height of the interface at site x) an energy $H_\Lambda(\varphi)$ is assigned, where the Hamiltonian is assumed to be of gradient type,

$$H_\Lambda(\varphi) = \sum_{x \in \Lambda} \sum_{i=1}^d W(\nabla_i \varphi(x)) \quad \text{with} \quad \nabla_i \varphi(x) = \varphi(x + e_i) - \varphi(x).$$

We consider tilted boundary conditions, i.e., for $u \in \mathbb{R}^d$

$$\psi_u(x) = x \cdot u \quad \text{for} \quad x \in \partial\Lambda = \{z \in \mathbb{Z}^d \setminus \Lambda : |z - x| = 1 \text{ for some } x \in \Lambda\}.$$

The finite volume Gibbs measure for inverse temperature β is given by

$$\gamma_\Lambda^{\psi_u}(\mathrm{d}\varphi) = \frac{1}{Z_\Lambda^{\psi_u}} e^{-\beta H_\Lambda(\varphi)} \prod_{x \in \Lambda} \mathrm{d}\varphi(x) \prod_{x \in \partial\Lambda} \delta_{\psi_u(x)}(\mathrm{d}\varphi(x))$$

where $Z_\Lambda^{\psi_u}$ is the partition function which normalizes the measure. Later it will be more convenient to work with periodic boundary condition rather than Dirichlet boundary condition because the problem then remains translation invariant. Imposing a tilt u corresponds

to working with functions such that $x \mapsto \varphi - x \cdot u$ is periodic. This can be reduced to the study of periodic functions by replacing the expression $W(\nabla_i \varphi(x))$ in the Hamiltonian by $W(\nabla_i \varphi(x) + u_i)$, see (2.4) below. For the equivalence of various ways of imposing a tilt, at least on the level of the thermodynamic limit of the free energy, see [KL14].

In the case of strictly convex, symmetric W a lot is known: The infinite volume gradient Gibbs measure exists and is uniquely determined by the tilt [FS97]. The long distance behaviour is described by the Gaussian free field (see [NS97] and [GOS01]) and the decay of the covariance is polynomial as in the massless Gaussian case [DD05]. Moreover the surface tension is strictly convex [DGI00]. A nice overview of these results and the used techniques can be found in [Fun05] or [Vel06].

Much less is known for models with non-convex potentials. At moderate temperature and zero tilt Biskup and Kotecký showed in [BK07] the existence of two ergodic infinite volume Gibbs measures for a particular non-convex potential, a mixture of two centered Gaussians. For this potential it can nevertheless be shown that both gradient Gibbs measures scale to a Gaussian free field [BS11].

The high temperature regime of potentials of the form

$$W(\eta) = W_0(\eta) + g_0(\eta), \quad W_0 \text{ strictly convex}, \quad \sqrt{\beta} \|g\|_{L^1} \text{ small}$$

is analyzed in [CDM09] and [CD12]. The authors prove strict convexity of the surface tension, uniqueness of the ergodic gradient Gibbs measure, scaling to the Gaussian free field and polynomial decay of the covariance for any tilt. Note that smallness of $\sqrt{\beta} \|g\|_{L^1}$ still allows for non-convex W .

For low temperatures a finite range decomposition of the Gaussian measure and renormalization group techniques in the spirit of [Bry09] can be used to get first results for potentials which are small non-convex perturbations of the quadratic potential, i.e.,

$$W(\eta) = \frac{1}{2} \eta^2 + V(\eta), \quad V \text{ small, but } W \text{ nonconvex.}$$

In [AKM] a representation for the finite volume Gibbs measure is obtained and is applied there to show strict convexity of the surface tension for small tilt u .

The objective of this note is to use the results of [AKM] to prove a Central Limit Theorem for these models and to show that their behaviour at long distances is governed by a suitable Gaussian free field. On a slightly finer scale we also prove convergence of the covariances of this model.

Note that in estimates a constant is always denoted by C but can change from line to line.

2 Setting and Results

2.1 Setting

Let the potential $W : \mathbb{R} \rightarrow \mathbb{R}$ be a perturbation of the quadratic potential,

$$W(\eta) = \frac{1}{2}\eta^2 + V(\eta), \quad V : \mathbb{R} \rightarrow \mathbb{R}. \quad (2.1)$$

Following [FS97], we enforce tilted boundary conditions and simultaneously shift invariance by considering fields on the torus and a shifted potential $W(\cdot + u_i)$: For $L > 0$ a large fixed odd integer consider the discrete torus $\mathbb{T}_N^d = (\mathbb{Z}/L^N\mathbb{Z})^d$ of side length L^N . This torus can be represented by the cube Λ_N of length L^N ,

$$\Lambda_N := \left\{ x \in \mathbb{Z}^d : |x|_\infty \leq \frac{1}{2}(L^N - 1) \right\}, \quad (2.2)$$

equipped with the metric $d_{\text{per}}(x, y) := |x - y|_{\text{per}} := \inf \left\{ |x - y + k|_\infty : k \in (L^N\mathbb{Z})^d \right\}$ where $|x|_\infty := \max_{i=1, \dots, d} |x_i|$.

Due to the gradient type of the Hamiltonian we restrict to fields with mean value zero,

$$\chi_N := \left\{ \varphi : \Lambda_N \rightarrow \mathbb{R}, \sum_{x \in \Lambda_N} \varphi(x) = 0 \right\}, \quad (2.3)$$

equipped with the scalar product $(\varphi, \psi) = \sum_{x \in \Lambda_N} \varphi(x)\psi(x)$.

By (2.1) the shifted Hamiltonian can be written as

$$H_N(\varphi) = \mathcal{E}_N(\varphi) + \frac{1}{2}L^{Nd}|u|^2 + \sum_{x \in \Lambda_N} \sum_{i=1}^d V(\nabla_i \varphi(x) + u_i), \quad (2.4)$$

where

$$\mathcal{E}_N(\varphi) := \frac{1}{2}(\mathcal{A}^0 \varphi, \varphi) = \frac{1}{2} \sum_{x \in \Lambda_N} \sum_{i=1}^d (\nabla_i \varphi(x))^2, \quad \mathcal{A}^0 = \sum_{i,j} \delta_{ij} \nabla_j^* \nabla_i$$

with the adjoint difference operator $\nabla_i^* \varphi(x) = \varphi(x - e_i) - \varphi(x)$. The finite volume Gibbs measure is

$$\nu_{H_N}(d\varphi) = \frac{1}{Z_{H_N}} e^{-\beta H_N(\varphi)} d\lambda_N(\varphi) \quad (2.5)$$

where $Z_{H_N} = \int_{\chi_N} e^{-\beta H_N(\varphi)} d\lambda_N(\varphi)$ is the partition function and $d\lambda_N$ denotes the $(L^{Nd}-1)$ -dimensional Hausdorff measure on χ_N .

Remark. To simplify the notation we set $\beta = 1$ in the following. Note further that we do not make explicit the dependence on u in the notation since it plays no special role here.

2.2 Results

For computing the scaling limit we define, for $f \in C_c^\infty(\mathbb{T}^d; \mathbb{R}^d)$,

$$f^N(x) := L^{-N\frac{d}{2}} f(L^{-N}x) \quad (2.6)$$

and introduce the slowly varying scaled field

$$\nabla\varphi(f^N) := (\nabla\varphi, f^N) \text{ for } x \in \Lambda_N. \quad (2.7)$$

We describe the distribution of this random vector by the Laplace transform

$$\int e^{-(\nabla\varphi, f^N)} \nu_{H_N}(d\varphi) \quad \text{for } f \in \mathbb{R}^{\Lambda_N}. \quad (2.8)$$

We are interested in the existence of some limiting distribution $(\nabla\varphi)^{\text{scale}}$ which is in general a random field on distributions. Then the limiting distribution, if it exists, is the joint distribution ν^{scale} for generalized gradient random fields $((\nabla\varphi)^{\text{scale}}(f))_{f \in C_c^\infty(\mathbb{T}^d)}$, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, such that

$$\mathbb{E}_{\nu^{\text{scale}}} \left(e^{-\nabla\varphi^{\text{scale}}(f)} \right) = \lim_{N \rightarrow \infty} \int_{\chi_N} e^{-(\nabla\varphi, f^N)} \nu_{H_N}(d\varphi) \quad \text{for all } f \in C_c^\infty(\mathbb{T}^d).$$

We will show that for small initial perturbation of the quadratic potential the measure ν_{H_N} tends to a continuum Gaussian free field with a renormalized covariance in the sense of convergence of Laplace transforms of the measures.

For stating the smallness condition on V we introduce the second order Taylor remainder U of V ,

$$U(s, t) = V(s + t) - V(t) - V'(t)s,$$

and set, for $z \in \mathbb{R}^d$,

$$\mathcal{K}(z) = e^{-\sum_{i=1}^d U(z_i, u_i)} - 1. \quad (2.9)$$

We define, for a multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_i \in \mathbb{N}$, the length $|\alpha| = \sum_{i=1}^d \alpha_i$ and the operator $\partial^\alpha = \prod_{i=1}^d \partial_i^{\alpha_i}$ with $\partial_i^0 := 1$. For $\zeta > 0$ and $r_0 \in \mathbb{N}$ we define the norm

$$\|\mathcal{K}\|_\zeta = \sup_{z \in \mathbb{R}^d} \sum_{|\alpha| \leq r_0} \zeta^{|\alpha|} |\partial_z^\alpha \mathcal{K}(z)| e^{-\zeta^2 |z|^2}. \quad (2.10)$$

Theorem 2.1

There is $\zeta > 0$, $r_0 \in \mathbb{N}$ and $\rho > 0$ such that for all V with $\|\mathcal{K}\|_\zeta \leq \rho$ there is $(\bar{q}_{ij})_{i,j \in \{1, \dots, d\}}$ satisfying for any $f \in C_c^\infty(\mathbb{T}^d; \mathbb{R}^d)$ on a subsequence

$$\int e^{-(\nabla\varphi, f^N)} \nu_{H_N}(d\varphi) \rightarrow e^{\frac{1}{2}(\text{div } f, \mathcal{C} \text{ div } f)_{L^2(\mathbb{T}^d)}}$$

as N tends to infinity, where the right hand side is the Laplace transform of the continuum Gaussian free field on \mathbb{T}^d with covariance $\mathcal{C} = (\mathcal{A})^{-1}$,

$$\mathcal{A} := - \sum_{i,j=1}^d (\delta_{ij} + \bar{q}_{ij}) \partial_j \partial_i.$$

Remark. This is a Central Limit Theorem for strongly dependent random fields. Indeed, let f be an approximation of the characteristic function of $Q(0) = (-\frac{1}{2}; \frac{1}{2})^d$. By the above Theorem the limiting distribution of

$$\phi^N := L^{-N\frac{d}{2}} \sum_{x \in \Lambda_N} \nabla \varphi(x)$$

is Gaussian. This also explains the choice of the scaling factor used here which is the typical one in Central Limit Theorems. The classical Central Limit Theorem cannot be applied due to the long-range gradient-gradient correlations of the measure.

Theorem 2.1 captures the limiting behaviour if we average over the whole system of scale L^{dN} and rescale. If we are not interested in the full distribution but only in covariances we can also show convergence to the covariances of a Gaussian measure if we only average over $L^{\alpha d N}$ many points, $\alpha < 1$. To state this result precisely fix $a, b \in \mathbb{R}^d$ and, for $z = a, b$, let $J_z \in C_c^\infty(Q(z); \mathbb{R}^d)$, where $Q(z) = z + (-\frac{1}{2}, \frac{1}{2})^d$. Define for $0 < \alpha \leq 1$

$$J_z^N(x) := L^{-\alpha N \frac{d}{2}} J_z \left(\frac{x}{L^{\alpha N}} \right) \quad \text{for } x \in \Lambda_N. \quad (2.11)$$

Theorem 2.2

For ζ, r_0, ρ, V and \bar{q} as in Theorem 2.1 there is $\alpha_0 < 1$ such that for any $\alpha \in [\alpha_0; 1]$ and for all $J_z \in C_c^\infty(Q(z); \mathbb{R}^d)$ on a subsequence

$$\text{Cov}_{\nu_{H_N}}((\nabla \varphi, J_a^N), (\nabla \varphi, J_b^N)) \rightarrow (\text{div}^* J_a, \mathcal{C} \text{div}^* J_b)_{L^2(\mathbb{R}^d)}$$

as N tends to infinity, where the right hand side is the covariance of the continuum Gaussian free field on \mathbb{R}^d , i.e., \mathcal{C} is the inverse of the operator $\sum_{i,j=1}^d (\delta_{ij} + \bar{q}_{ij}) \partial_j \partial_i$ defined on functions on \mathbb{R}^d .

3 Outline of the Proofs

The proofs of Theorem 2.1 and 2.2 rely on a representation for the finite volume Gibbs measure constructed in [AKM]. There, the measure ν_{H_N} is written as perturbation of a Gaussian measure μ^q , i.e., $\nu_{H_N} = F_0^q(\varphi) \mu^q$. In [AKM13] μ^q is decomposed into Gaussian

measures $\mu_1^q, \dots, \mu_{N+1}^q$ with a suitable finite range property such that $\mu^q = \mu_1^q * \dots * \mu_{N+1}^q$. This yields

$$\begin{aligned} \int F_0^q(\varphi) \mu^q(d\varphi) &= \int F_0^q(\varphi) \mu_1^q * \dots * \mu_{N+1}^q(d\varphi) = \int F_1^q(\varphi) \mu_2^q * \dots * \mu_{N+1}^q(d\varphi) \\ &= \dots = \int F_N^q(\varphi) \mu_{N+1}^q(d\varphi). \end{aligned} \quad (3.1)$$

It can be shown that the effective interactions F_k^q are expressed as the composition of a 'relevant' term $e^{-H_k^q(\varphi)}$, where $H_k^q(\varphi)$ is quadratic and local in $\nabla\varphi$, and an 'irrelevant' term $K_k^q(\varphi)$ such that the map $K_k \mapsto K_{k+1}$ is a contraction in suitable norms. From this contraction property one can deduce a Stable Manifold Theorem for the evolution of the (H_k^q, K_k^q) variables. This gives the possibility to choose q such that the initial perturbation F_0^q lies on the stable manifold. Hence after N steps a nice representation, which corresponds to $H_N^q = 0$, is obtained.

In the following we first give a precise statement of corresponding result in [AKM] and collect useful consequences. Then we give a sketch of the proof of Theorem 2.1 and Theorem 2.2. In the next Section we provide the detailed proofs.

3.1 Representation for ν_{H_N}

To state the result in [AKM] we have to introduce some objects. First of all we rewrite the Gibbs measure $\nu_{H_N}(d\varphi)$ as perturbation of a Gaussian measure μ^q , $q \in \mathbb{R}_{\text{sym}}^{d \times d}$, with covariance $\mathcal{C}^q = (\mathcal{A}^q)^{-1}$, where

$$\mathcal{A}^q = \sum_{i,j} (\delta_{ij} + q_{ij}) \nabla_j^* \nabla_i. \quad (3.2)$$

For this recall that U is the second order Taylor remainder of V , i.e.,

$$U(s, t) = V(s + t) - V(t) - V'(t)s \quad (3.3)$$

and insert artificially the so called fine-tuning parameters $\frac{1}{2}(q\nabla\varphi, \nabla\varphi)$ and $\lambda^q \in \mathbb{R}$ to get

$$e^{-H_N(\varphi)} d\lambda_N(\varphi) = \kappa F(\varphi) \mu^q(d\varphi) \quad (3.4)$$

where $\kappa = e^{-\frac{1}{2}L^{Nd}|u|^2} e^{-L^{Nd}\sum_i V(u_i)} e^{-\lambda^q L^{Nd} Z_N^q}$ and

$$F(\varphi) = e^{\frac{1}{2}(q\nabla\varphi, \nabla\varphi) + \lambda^q L^{Nd}} e^{-\sum_x \sum_i U(\nabla_i \varphi(x), u_i)}. \quad (3.5)$$

Here Z_N^q is the partition function of the measure μ^q . Then

$$\nu_{H_N}(d\varphi) = F(\varphi) \mu^q(d\varphi) \frac{1}{\int F(\varphi) \mu^q(d\varphi)}. \quad (3.6)$$

Furthermore we need the norm on the irrelevant part used in the last integration step in (3.1). In the following several constants will appear which are needed for the construction in [AKM] but they will not be explained or motivated here.

First we define a norm on fields $\varphi \in \mathbb{R}^{\Lambda_N}$ by

$$|\varphi|_{N,\Lambda_N} = \max_{1 \leq s \leq 3} \sup_{x \in \Lambda_N} \frac{1}{h} L^{N(\frac{d-2}{2}+s)} |\nabla^s \varphi(x)| \quad (3.7)$$

where $|\nabla^s \varphi(x)|^2 = \sum_{|\alpha|=s} |\nabla^\alpha \varphi(x)|^2$, α is a multiindex and $h > 0$. We introduce the quantities

$$G_{N,x}(\varphi) = \frac{1}{h^2} \left(|\nabla \varphi(x)|^2 + L^{2N} |\nabla^2 \varphi(x)|^2 + L^{4N} |\nabla^3 \varphi(x)|^2 \right) \quad (3.8)$$

$$\text{and } g_{N,x}(\varphi) = \frac{1}{h^2} \sum_{s=2}^4 L^{(2s-2)N} \sup_{y \in \Lambda_N} |\nabla^s \varphi(y)|^2 \quad (3.9)$$

which are used to define the so-called large field regulator

$$w_N^{\Lambda_N}(\varphi) = e^{\sum_{x \in \Lambda_N} \omega(2^d g_{N,x}(\varphi) + G_{N,x}(\varphi))}, \quad \omega \in \mathbb{R}. \quad (3.10)$$

Next we determine a seminorm which controls the Taylor remainder of a function F on fields,

$$|F(\varphi)|^{N,\Lambda_N} = \sum_{s=0}^{r_0} \frac{1}{s!} \sup_{|\bar{\varphi}|_{N,\Lambda_N} \leq 1} |D^s F(\varphi)(\bar{\varphi}, \dots, \bar{\varphi})| \quad (3.11)$$

for some $r_0 > 0$. Finally we define the norm

$$\|F\|_{N,\Lambda_N} = \sup_{\varphi} |F(\varphi)|^{N,\Lambda_N} w_N^{-\Lambda_N}(\varphi). \quad (3.12)$$

Note that, in comparison to [AKM], we skip any dependencies of maps on subsets $X \subset \Lambda_N$ since we do not need it here.

Let

$$\mu^q = \mu_1^q * \dots * \mu_{N+1}^q$$

be a decomposition of μ^q into Gaussian measures with range on increasing blocks, see [AKM13] for an exact definition and for existence of such a decomposition whenever q is small enough, i.e., $\|q\| \leq \frac{1}{2}$ where the norm $\|q\|$ is the operator norm of q viewed as operator on \mathbb{R}^d equipped with the metric $|q|_{l_2} = (\sum_i |q_i|^2)^{\frac{1}{2}}$.

In the following Proposition we use the notation

$$F * \mu(\xi) = \int F(\varphi + \xi) \mu(d\varphi).$$

Proposition 3.1

There exist positive constants ρ , $\rho_1 \leq \frac{1}{2}$, ζ and $\eta \in (0, 1)$, η independent on N , such that for suitable chosen constants L , h , ω and r_0 and for any \mathcal{K} with $\|\mathcal{K}\|_\zeta \leq \rho$ there is a parameter $q = q(\mathcal{K}, N)$ with $\|q\| \leq \rho_1$ satisfying

$$F * \mu^q(\xi) = 1 + \int K_N(\varphi + \xi) \mu_{N+1}^q(d\varphi) \quad (3.13)$$

such that

$$\|K_N\|_{N, \Lambda_N} \leq C \eta^N. \quad (3.14)$$

A choice for L , h , ω and r_0 is made in [AKM], Proposition 4.6. The existence of the constants ρ , ρ_1 and ζ and of the parameter q and the formula (3.13) can be found in [AKM], Theorem 4.9. The exponential decay of the norm (3.14) is a consequence of Proposition 8.1 and of the construction of the corresponding Banachspace in Subsection 4.5, both in [AKM].

From the results in [AKM] one can also deduce the following estimate on maps F on fields.

Lemma 3.2

For any $s \leq r_0$ it holds

$$\begin{aligned} & \left| \int D^s F(\varphi + \xi)(\xi_1, \dots, \xi_s) \mu_{N+1}^q(d\varphi) \right| \\ & \leq C |\xi_1|_{N, \Lambda_N} \cdots |\xi_s|_{N, \Lambda_N} \|F\|_{N, \Lambda_N} \left(w_N^{\Lambda_N}(\xi) \right)^2. \end{aligned}$$

Proof. By looking carefully at the definition of the norm $\|\cdot\|_{N, \Lambda_N}$ one can easily obtain the following estimate:

$$\begin{aligned} & \left| \int D^s F(\varphi + \xi)(\xi_1, \dots, \xi_s) \mu_{N+1}^q(d\varphi) \right| \\ & \leq C |\xi_1|_{N, \Lambda_N} \cdots |\xi_s|_{N, \Lambda_N} \|F\|_{N, \Lambda_N} \int w_N^{\Lambda_N}(\varphi + \xi) \mu_{N+1}^q(d\varphi). \end{aligned}$$

An adjustment of the proof of Lemma 5.2 in [AKM] (there the case $k = N$ is excluded) gives

$$\int w_N^{\Lambda_N}(\varphi + \xi) \mu_{N+1}^q(d\varphi) \leq C \left(w_N^{\Lambda_N}(\xi) \right)^2.$$

In fact the adjustment is a huge simplification since we do not have to deal with the boundary terms which the authors of [AKM] have to for the scales $k < N$. \square

3.2 Sketch of the proofs

In order to apply Proposition 3.1 for the computation of the scaling limit and the smoothed covariance we do the following *key calculation*: By completing the square and linear transformation we get for a Gaussian measure μ_C

$$\begin{aligned} \int e^{-(\varphi, f)} F(\varphi) \mu_C(d\varphi) &= \frac{1}{Z} \int e^{-(\varphi, f)} e^{-\frac{1}{2}(\varphi, C^{-1}\varphi)} F(\varphi) d\varphi \\ &= e^{\frac{1}{2}(f, Cf)} \frac{1}{Z} \int e^{-\frac{1}{2}(\varphi + Cf, C^{-1}(\varphi + Cf))} F(\varphi) d\varphi \\ &= e^{\frac{1}{2}(f, Cf)} \int F(\varphi - Cf) \mu_C(d\varphi). \end{aligned} \quad (3.15)$$

Proof of Theorem 2.1, main ideas. We compute, using (3.6) and (3.15) and denoting $g^N = \sum_l \nabla_l^* f_l^N$,

$$\begin{aligned} \int e^{-(\nabla\varphi, f^N)} \nu_{H_N}(d\varphi) &= \frac{1}{\int F d\mu^q} \int e^{-(\varphi, g^N)} F(\varphi) \mu^q(d\varphi) \\ &= e^{\frac{1}{2}(g^N, C^q g^N)} \frac{1}{\int F d\mu^q} \int F(\varphi - C^q g^N) \mu^q(d\varphi) \\ &= e^{\frac{1}{2}(g^N, C^q g^N)} \frac{F * \mu^q(-C^q g^N)}{F * \mu^q(0)}. \end{aligned}$$

Now we apply the representation (3.13) in Proposition 3.1, Lemma 3.2 and the bound (3.14) in Proposition (3.1) to see that

$$|F * \mu^q(0) - 1| = \left| \int K_N(\varphi) \mu_{N+1}^q(d\varphi) \right| \leq C \|K_N\|_{N, \Lambda_N} w_N^{\Lambda_N}(0)^2 \leq C \eta^N w_N^{\Lambda_N}(0)^2.$$

By the same reasoning,

$$|F * \mu^q(-C^q g^N) - 1| \leq C \eta^N w_N^{\Lambda_N}(-C^q g^N)^2.$$

In Subsection 4.3, Lemma 4.7 we show that $w_N^{\Lambda_N}(0)$ and $w_N^{\Lambda_N}(-C^q g^N)$ can be bounded uniformly in N such that

$$\frac{F * \mu^q(-C^q g^N)}{F * \mu^q(0)} \rightarrow 1 \quad \text{as } N \rightarrow \infty.$$

The convergence

$$e^{\frac{1}{2}(g^N, C^q g^N)} \rightarrow e^{\frac{1}{2}(\operatorname{div} f, C \operatorname{div} f)_{L^2}} \quad \text{as } N \rightarrow \infty$$

is proved in detail in Subsection 4.2, see Proposition 4.5. \square

For the proof of Theorem 2.2 note that we can compute the covariance by taking derivatives of the logarithm of a generating functional. For a measure $\nu = \frac{1}{Z} e^{-H} d\lambda$ and random variables X and Y it holds

$$\text{Cov}_\nu(X, Y) = \partial_s \partial_t \Big|_{s=t=0} \ln \int e^{-(sX+tY)} e^{-H} d\lambda. \quad (3.16)$$

Proof of Theorem 2.2, main ideas. We use the representation of the covariance (3.16) as follows

$$\text{Cov}_{\nu_{H_N}}((\nabla \varphi, J_a^N), (\nabla \varphi, J_b^N)) = \partial_s \partial_t \Big|_{s=t=0} \ln \int e^{-(sJ_a^N + tJ_b^N, \nabla \varphi)} e^{-H_N(\varphi)} d\lambda_N(\varphi).$$

As in the proof of Theorem 2.1 we compute, using (3.4), (3.13) and (3.15) and denoting $g_z^N = \sum_l \nabla_l^* J_z^N$ for $z = a, b$,

$$\begin{aligned} & \text{Cov}_{\nu_{H_N}}((\nabla \varphi, J_a^N), (\nabla \varphi, J_b^N)) \\ &= \partial_s \partial_t \Big|_{s=t=0} \ln \left[\kappa e^{\frac{1}{2}(sg_a^N + tg_b^N, \mathcal{C}^q(sg_a^N + tg_b^N))} F * \mu^q(-s\mathcal{C}^q g_a^N - t\mathcal{C}^q g_b^N) \right] \\ &= (g_a^N, \mathcal{C}^q g_b^N) \\ & \quad + \frac{1}{F * \mu^q(0)} \int D^2 K_N(\varphi) (-\mathcal{C}^q g_a^N, -\mathcal{C}^q g_b^N) \mu_{N+1}^q(d\varphi) \\ & \quad - \frac{1}{F * \mu^q(0)^2} \int DK_N(\varphi) (-\mathcal{C}^q g_a^N) \mu_{N+1}^q(d\varphi) \int DK_N(\varphi) (-\mathcal{C}^q g_b^N) \mu_{N+1}^q(d\varphi). \end{aligned} \quad (3.17)$$

As in the proof of Theorem 2.1 it holds by the use of Proposition 3.1

$$|F * \mu^q(0) - 1| \leq C\eta^N w_N^{\Lambda_N}(0)^2.$$

and thus the denominators in (3.17) tend to 1 as N tends to infinity. Employing Lemma 3.2 and (3.14) in Proposition 3.1 we further get

$$\left| \int D^2 K_N(\varphi) (-\mathcal{C}^q g_a^N, -\mathcal{C}^q g_b^N) \mu_{N+1}^q(d\varphi) \right| \leq C\eta^N |\mathcal{C}^q g_a^N|_{N, \Lambda_N} |\mathcal{C}^q g_b^N|_{N, \Lambda_N}$$

and for $z = a, b$

$$\left| \int DK_N(\varphi) (-\mathcal{C}^q g_z^N) \mu_{N+1}^q(d\varphi) \right| \leq C\eta^N |\mathcal{C}^q g_z^N|_{N, \Lambda_N}.$$

Hence it remains to show

$$\eta^N |\mathcal{C}^q g_a^N|_{N, \Lambda_N} |\mathcal{C}^q g_b^N|_{N, \Lambda_N} \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (3.18)$$

The exponential decay η^N allows us to consider values $\alpha \leq 1$. In Lemma 4.8 we show that there exist a decreasing function τ satisfying $\tau(1) = 1$ such that

$$|\mathcal{C}^q g_z^N|_{N, \Lambda_N} \leq C \tau(\alpha)^N.$$

This gives the possibility to choose $\alpha_0 < 1$ as small as possible such that $\eta\tau(\alpha_0)^2 < 1$ and hence $(\eta\tau(\alpha)^2)^N \rightarrow 0$ as N tends to infinity for all $\alpha \in [\alpha_0, 1]$.

Then it only remains to show

$$(g_a^N, \mathcal{C}^q g_b^N) \rightarrow (\operatorname{div} J_a, \mathcal{C} \operatorname{div} J_b)_{L^2}.$$

This is done in Subsection 4.2, Proposition 4.6. □

Remark. For higher moments (say of order s) one has to choose α_0 such that

$$\eta\tau(\alpha_0)^s < 1$$

which implies $\alpha_0 > 1 - \frac{-\ln \eta}{s(\frac{d}{2}+4)}$ (see the formula for $\tau(\alpha)$ in Lemma 4.8) so $\alpha_0 \rightarrow 1$ as $s \rightarrow \infty$.

Estimates of correlation functions can be done in more generality by inserting external fields into the partition function and extending the flow of (H_k^q, K_k^q) to these observable variables, compare [BBS14]. This amounts to extending norm estimates in [AKM] to the case of included variables and computing explicitly the flow of the observables. We plan to pursue this in future work.

4 Details of the Proofs

4.1 Scaled discrete setting

For the proof of the convergence of \mathcal{C}^q to \mathcal{C} we switch to a scaled setting. Let $0 < \alpha \leq 1$. Define

$$\mathcal{A}_N = \sum_{i,j} a_{ij}^N D_{N,j}^* D_{N,i} \quad \text{with } a_{ij}^N = \delta_{ij} + q_{ij}^N$$

where

$$D_{N,j}\varphi(x) = \frac{\varphi(x + L^{-\alpha N} e_j) - \varphi(x)}{L^{-\alpha N}} \quad \text{and} \quad D_{N,j}^*\varphi(x) = \frac{\varphi(x - L^{-\alpha N} e_j) - \varphi(x)}{L^{-\alpha N}}.$$

We will also use

$$\operatorname{div}_N^* h(x) = \sum_l D_{N,l}^* h_l(x) \quad \text{for } h : \Lambda_N \rightarrow \mathbb{R}^d. \quad (4.1)$$

Further define the scaled discrete torus

$$\Lambda'_N = \Lambda_N / L^{\alpha N}$$

of spacing $L^{\alpha N}$ with fundamental domain embedded into $\left[-\frac{L^{(1-\alpha)N}}{2}, \frac{L^{(1-\alpha)N}}{2}\right]^d$. The corresponding torus in continuum is thus $\mathbb{T}_{R_N}^d$ for $R_N = L^{(1-\alpha)N}$.

For maps $\varphi, \psi : \Lambda'_N \rightarrow \mathbb{R}$ define

$$\begin{aligned}\langle \varphi, \psi \rangle_{l^2} &:= L^{-\alpha Nd} \sum_{x \in \Lambda'_N} \varphi(x) \psi(x), \\ \langle \varphi, \psi \rangle_{w^{1,2}} &:= \langle \varphi, \psi \rangle_{l^2} + L^{-\alpha Nd} \sum_{x \in \Lambda'_N} \sum_{k=1}^d D_{N,k} \varphi(x) D_{N,k} \psi(x).\end{aligned}$$

Further, let

$$\chi'_N = \left\{ \varphi : \Lambda'_N \rightarrow \mathbb{R}, \sum_{x \in \Lambda'_N} \varphi(x) = 0 \right\}$$

which becomes a Hilbert space when endowed with the scalar product $\langle \cdot, \cdot \rangle_{w^{1,2}}$.

For convenience of the reader we include a proof of the following standard result.

Proposition 4.1

For $g : \Lambda'_N \rightarrow \mathbb{R}$ there is a unique weak solution $u_N \in \chi'_N$ to

$$\mathcal{A}_N u_N = D_{N,l}^* g \quad \text{in } \Lambda'_N.$$

This solution satisfies

$$\|u_N\|_{w^{1,2}} \leq C L^{2N(1-\alpha)} \|g\|_{l^2}.$$

Moreover, there is a constant independent of N such that

$$\|D_N u_N\|_{l^2} \leq C \|g\|_{l^2}.$$

Proof. This works as in the continuum. Define

$$\mathcal{L}_N(\varphi, \psi) := L^{-\alpha Nd} \sum_{x \in \Lambda'_N} \sum_{i,j} a_{ij}^N D_{N,i} \varphi(x) D_{N,j} \psi(x)$$

and

$$F(\psi) := L^{-\alpha Nd} \sum_{x \in \Lambda'_N} g(x) D_{N,l} \psi(x).$$

In χ'_N a Poincaré inequality holds (can be found, e.g., in [BGM04], Lemma B.2) and thus \mathcal{L}_N is a continuous coercive bilinear form. Indeed, coercivity follows from

$$\mathcal{L}_N(\psi, \psi) \geq \lambda_{\min}^N \sum_k \|D_{N,k} \psi\|_{l^2}^2, \quad (4.2)$$

where we denote by λ_{\min}^N the smallest eigenvalue of a_{ij}^N . By the Poincaré inequality it holds

$$\|\psi\|_{w^{1,2}}^2 = \|\psi\|_{l^2}^2 + \sum_k \|D_{N,k}\psi\|_{l^2}^2 \leq (1 + C(\Lambda'_N)) \sum_k \|D_{N,k}\psi\|_{l^2}^2.$$

where $C(\Lambda'_N) = C_d L^{2N(1-\alpha)}$. Hence

$$\mathcal{L}_N(\psi, \psi) \geq \frac{\lambda_{\min}^N}{1 + C(\Lambda'_N)} \|\psi\|_{w^{1,2}}^2.$$

Further, F is an element of the dual space by the estimate

$$|F(\psi)| \leq \|g\|_{l^2} \sum_k \|D_{N,k}\psi\|_{l^2} \leq \|g\|_{l^2} \|\psi\|_{w^{1,2}}. \quad (4.3)$$

Thus the Lax-Milgram Theorem provides a unique solution $u_N \in \chi'_N$ of $\mathcal{L}_N(u_N, \psi) = F(\psi)$ for all $\psi \in \chi'_N$ together with the estimate

$$\|u_N\|_{w^{1,2}} \leq \frac{1 + C(\Lambda'_N)}{\lambda_{\min}^N} \|F\|_{(\chi'_N)^*} \leq \frac{1 + C}{\lambda_{\min}^N} L^{2(1-\alpha)N} \|g\|_{l^2}.$$

For the estimate which is independent on N we use (4.3) with $\psi = u_N$ and (4.2) to obtain

$$\begin{aligned} \|D_N u_N\|_{l^2}^2 &= \sum_k \|D_{N,k} u_N\|_{l^2}^2 \leq \frac{1}{\lambda_{\min}^N} |\mathcal{L}_N(u_N, u_N)| = \frac{1}{\lambda_{\min}^N} |F(u_N)| \\ &\leq \frac{1}{\lambda_{\min}^N} \|g\|_{l^2} \sum_k \|D_{N,k} u_N\|_{l^2}. \end{aligned}$$

Finally note that by the convergence of q^N to \bar{q} (see Lemma 4.4) λ_{\min}^N can be bounded from below by $\lambda_{\min} - \epsilon > 0$, λ_{\min} being the minimal eigenvalue of (a_{ij}) . \square

Corollary 4.2

Let u_N be the solution in Proposition 4.1 and $g \in C_c^\infty$. Then

$$\sup_{x \in \Lambda'_N} |u_N(x)| \leq C L^{2(1-\alpha)N} \|g\|_{C^d}.$$

Proof. By discrete differentiation of the strong form of the equation one obtains $\mathcal{A}_N D_N^s u_N = D_{N,l}^* D_N^s g$ for any multiindex s . We apply Proposition 4.1 to get

$$\|D_N^s u_N\|_{w^{1,2}} \leq C L^{2N(1-\alpha)} \|D_N^s g\|_{l^2}.$$

Now we use the discrete Sobolev embedding (see, e.g., [BGM04], Lemma B.1) to obtain

$$\sup_x |u_N(x)| \leq C_d \sum_{|s| \leq d} \|D_N^s u_N\|_{l^2}.$$

This gives the desired estimate. \square

We now transform the terms of interest from the discrete unscaled into the discrete scaled setting.

Lemma 4.3

1. The kernels C_N^q and C_N of the inverses of the original operator \mathcal{A}^q and the scaled operator \mathcal{A}_N are related as follows:

$$C_N(x') = L^{\alpha N(d-2)} C_N^q(L^{\alpha N} x')$$

and thus

$$(\nabla_k C_N^q \nabla_l^* g^N)(x) = (D_{N,l} C_N D_{N,l}^* g)(L^{-\alpha N} x) \quad (4.4)$$

for $g^N(x) = L^{-\alpha N \frac{d}{2}} g(L^{-\alpha N} x)$.

2. Let $f \in C_c^\infty(Q(z_1)), g \in C_c^\infty(Q(z_2)), z_1, z_2 \in \mathbb{R}^d, 0 < \alpha \leq 1$ and define, for $x \in \Lambda_N$,

$$f^N(x) = L^{-\alpha N \frac{d}{2}} f(L^{-\alpha N} x) \quad \text{and} \quad g^N(x) = L^{-\alpha N \frac{d}{2}} g(L^{-\alpha N} x).$$

Then

$$(\nabla_l^* f^N, C_N^q \nabla_l^* g^N) = \langle D_{N,l}^* f, C_N D_{N,l}^* g \rangle_{l^2}.$$

Proof. 1. Fix any discrete function $w \in \chi_N$ and let $u \in \chi_N$ be a unique solution to $\mathcal{A}^q u(x) = w(x)$ for $x \in \Lambda_N$ and $v' \in \chi'_N$ be a unique solution of $\mathcal{A}_N v'(x') = w(L^{\alpha N} x')$ for $x' \in \Lambda'_N$. Then

$$\begin{aligned} w(x) &= \mathcal{A}^q u(x) = \sum_{i,j} a_{ij}^N \nabla_j^* \nabla_i u(x) = L^{-2\alpha N} \sum_{i,j} a_{ij}^N D_{N,j}^* D_{N,i} u(L^{\alpha N} x') \\ &= L^{-2\alpha N} \mathcal{A}_N u(L^{\alpha N} x') \end{aligned}$$

and uniqueness of the solution implies $v'(x') = L^{-2\alpha N} u(L^{\alpha N} x')$. When writing the solutions in terms of the inverse operator kernels C_N^q and C_N , we get

$$\begin{aligned} v'(x') &= L^{-\alpha N d} \sum_{y' \in \Lambda'_N} C_N(x' - y') w(L^{\alpha N} y') \quad \text{and} \\ L^{-2\alpha N} u(L^{\alpha N} x') &= L^{-2\alpha N} \sum_{y \in \Lambda_N} C_N^q(L^{\alpha N} x' - y) w(y) \\ &= L^{-\alpha N d} \sum_{y' \in \Lambda'_N} L^{\alpha N d} L^{-2\alpha N} C_N^q(L^{\alpha N} (x' - y')) w(L^{\alpha N} y'). \end{aligned}$$

Hence the claim follows.

2. Use the first part of this lemma and insert definitions and scalings.

□

4.2 Convergence of the operators

Since the fine-tuning parameter q^N obtained in Proposition 3.1 is uniformly bounded by $1/2$ we get the following convergence result.

Lemma 4.4

There exist $\bar{q} \in \mathbb{R}_{sym}^{d \times d}$ and a subsequence $(N_k)_k$ such that

$$\|q^{N_k} - \bar{q}\| \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Define the elliptic differential operator

$$\mathcal{A} := \sum_{i,j} a_{ij} \partial_j^* \partial_i$$

where we use the convention that $\partial_j^* = -\partial_j$. Furthermore let $\text{div}^* := -\text{div}$. Remark that by the Lax-Milgram Theorem there is a (unique up to the addition of constants) solution $u \in W^{1,2}(\mathbb{T}^d)$ such that for $g \in L^2(\mathbb{T}^d)$ we have $\mathcal{A}u = \text{div}^* g$ in \mathbb{T}^d and a (unique up to the addition of constants) solution $u \in W_{\text{loc}}^{1,2}(\mathbb{R}^d)$ with $Du \in L^2(\mathbb{R}^d)$ such that for $g \in L^2(\mathbb{R}^d)$ we have $\mathcal{A}u = \text{div}^* g$ in \mathbb{R}^d .

Proposition 4.5

For $f \in L^2(\mathbb{T}^d; \mathbb{R}^d)$ and the corresponding scaled function f^N as defined in (2.6) it holds on a subsequence

$$\left(\sum_l \nabla_l^* f_l^N, \mathcal{C}^q \sum_l \nabla_l^* f_l^N \right) \rightarrow (f, DC \text{div}^* f)_{L^2(\mathbb{T}^d; \mathbb{R}^d)}.$$

Proof. We first apply Lemma 4.3 with $\alpha = 1$ to switch to the scaled setting,

$$\left(\sum_l \nabla_l^* f_l^N, \mathcal{C}^q \sum_l \nabla_l^* f_l^N \right) = \langle \text{div}_N^* f, \mathcal{C}_N \text{div}_N^* f \rangle_{l^2}.$$

Let $u_N = \mathcal{C}_N \text{div}_N^* f \in \chi'_N$ be the unique solution to $\mathcal{A}_N u_N = \text{div}_N^* f_N$ (see Proposition 4.1) with $f_N := f|_{\Lambda'_N}$.

Extend u_N and f_N to piecewise constant functions on the continuous torus \mathbb{T}^d . Divide \mathbb{T}^d into cubes Q_x of side length L^{-N} with centres $x \in \Lambda'_N$ and define $u_N(y) = u_N(x)$ for all $y \in Q_x$. It follows from the definition of the extension (recall also the definition of div_N^* in (4.1)) that

$$\langle \text{div}_N^* f_N, u_N \rangle_{l^2} = (\text{div}_N^* f_N, u_N)_{L^2(\mathbb{T}^d)} = (f_N, D_N u_N)_{L^2(\mathbb{T}^d; \mathbb{R}^d)}.$$

Let $u \in W^{1,2}(\mathbb{T}^d)$ be the solution (unique up to the addition of constants) to $\mathcal{A}u = \operatorname{div}^* f$ on \mathbb{T}^d . From Proposition 4.1 it follows that $D_N u_N$ is uniformly bounded in $L^2(\mathbb{T}^d)$. Below in Steps 1-3 we show that on a subsequence $D_N u_N \rightharpoonup Du$ in $L^2(\mathbb{T}^d)$. Then

$$\begin{aligned} & \left(\sum_l \nabla_l^* f_l^N, \mathcal{C}^q \sum_l \nabla_l^* f_l^N \right) - (f, DC \operatorname{div}^* f)_{L^2(\mathbb{T}^d; \mathbb{R}^d)} \\ &= (f_N, D_N u_N)_{L^2(\mathbb{T}^d; \mathbb{R}^d)} - (f, Du)_{L^2(\mathbb{T}^d; \mathbb{R}^d)} \\ &= (f_N - f, D_N u_N)_{L^2(\mathbb{T}^d; \mathbb{R}^d)} + (f, D_N u_N - Du)_{L^2(\mathbb{T}^d; \mathbb{R}^d)} \\ &\leq \|f_N - f\|_{L^2} \|D_N u_N\|_{L^2} + (f, D_N u_N - Du)_{L^2}. \end{aligned}$$

By construction of f_N and by the bound on $D_N u_N$ the first term tends to zero as $N \rightarrow \infty$ and by the weak convergence of $D_N u_N$ this also holds for the second term. Thus the claim follows.

Step 1: From the bound on $D_N u_N$ we get existence of $v \in L^2(\mathbb{T}^d; \mathbb{R}^d)$ such that on a subsequence N_k

$$D_{N_k} u_{N_k} \rightharpoonup v \text{ in } L^2(\mathbb{T}^d; \mathbb{R}^d).$$

Step 2: There is $u \in L^2(\mathbb{T}^d)$ such that $v = Du$ (in the sense of weak derivatives).

Proof: Let $\bar{u}_N := \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} u_N dx$. We use the discrete Poincaré inequality and Step 1 to get

$$\|u_N - \bar{u}_N\|_{L^2} \leq C \|D_N u_N\|_{L^2} \leq C.$$

Thus there is a subsequence (not denoted explicitly in the following) and $u \in L^2(\mathbb{T}^d)$ such that $u_N - \bar{u}_N \rightharpoonup u$ in $L^2(\mathbb{T}^d)$.

We take $\varphi \in W^{1,2}(\mathbb{T}^d; \mathbb{R}^d)$ to obtain the following convergence as $N \rightarrow \infty$:

$$\int D_N u_N \cdot \varphi dx = \int (u_N - \bar{u}_N) \operatorname{div}_N^* \varphi dx \rightarrow \int u \operatorname{div}^* \varphi dx = \int Du \cdot \varphi dx.$$

On the other hand we have by Step 1 for any $\varphi \in L^2(\mathbb{T}^d; \mathbb{R}^d)$

$$\int_{\mathbb{T}^d} D_N u_N \varphi dx \rightarrow \int_{\mathbb{T}^d} v \varphi dx$$

as N tends to infinity. Thus u is weakly differentiable and $Du = v$.

Step 3: The function u in Step 2 satisfies the equation $\mathcal{A}u = \operatorname{div}^* f$ and thus is unique up to the addition of constants.

Proof: For $\varphi \in C^1(\mathbb{T}^d)$ let φ_N be the function obtained by restriction to Λ'_N and piecewise constant extension to \mathbb{T}^d and insert φ_N into the weak form of the equation satisfied by u_N to obtain

$$\sum_{i,j} \int a_{i,j}^N D_{N,i} u_N D_{N,j} \varphi_N dx = \int f_N \cdot D_N \varphi_N dx.$$

Now $D_{N,j}\varphi_N$ converges uniformly to $\partial_j\varphi$. Hence the left hand side converges to $\sum_{i,j} \int a_{i,j} \partial_i u \partial_j \varphi \, dx$ and the right hand side converges to $\int f \cdot D\varphi \, dx$. Thus

$$\sum_{i,j} \int a_{i,j} \partial_i u \partial_j \varphi \, dx = \int f \cdot D\varphi \, dx$$

for all $\varphi \in C^1(\mathbb{T}^d)$. By density the identity holds for all $\varphi \in W^{1,2}(\mathbb{T}^d)$ and this finishes the proof. \square

The proof of the following Proposition is very similar to the proof of Proposition 4.5. We just have to take into account that in this case we have to work on increasing tori.

Proposition 4.6

For $J_a, J_b \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ with compact support in $Q(a)$ and $Q(b)$ respectively and the corresponding scaled functions J_a^N, J_b^N as defined in (2.11) it holds on a subsequence

$$\left(\sum_l \nabla_l^* J_a^N, \mathcal{C}^q \sum_l \nabla_l^* J_b^N \right) \rightarrow (J_a, D\mathcal{C} \operatorname{div}^* J_b)_{L^2(\mathbb{R}^d; \mathbb{R}^d)}.$$

Proof. First, apply Lemma 4.6 to switch to the scaled setting:

$$\left(\sum_l \nabla_l^* J_a^N, \mathcal{C}^q \sum_l \nabla_l^* J_b^N \right) = \langle \operatorname{div}_N^* J_a, \mathcal{C}_N \operatorname{div}_N^* J_b \rangle_{l_2}.$$

Let $u_N = \mathcal{C}_N \operatorname{div}_N^* J_b \in \chi'_N$ be the unique solution to $\mathcal{A}_N u_N = \operatorname{div}_N^*(J_b)_N$ (see Proposition 4.1), where $(J_b)_N := J_b|_{\Lambda'_N}$.

As in the proof of Proposition 4.5 we extend u_N and $(J_a)_N, (J_b)_N$ piecewise constant to the continuous torus $\mathbb{T}_{R_N}^d$ with $R_N = L^{(1-\alpha)N}$. It follows from the definition of the extension that

$$\langle \operatorname{div}_N^*(J_a)_N, u_N \rangle_{l_2} = (\operatorname{div}_N^*(J_a)_N, u_N)_{L^2(\mathbb{T}_{R_N}^d)} = ((J_a)_N, D_N u_N)_{L^2(\mathbb{T}_{R_N}^d; \mathbb{R}^d)}.$$

Let $u \in W_{\operatorname{loc}}^{1,2}(\mathbb{R}^d)$, $Du \in L^2(\mathbb{R}^d)$, be the solution (unique up to the addition of constants) to $\mathcal{A}u = \operatorname{div}^* J_b$ on \mathbb{R}^d . Let $I_N := \left(-\frac{R_N}{2}; \frac{R_N}{2}\right)$ be the fundamental domain of $\mathbb{T}_{R_N}^d$. We have a uniform bound on $\chi_{I_N} D_N u_N$ in $L^2(\mathbb{R}^d; \mathbb{R}^d)$ by Proposition 4.1 and as before we will show that on a subsequence $\chi_{I_N} D_N u_N \rightharpoonup Du$ in $L^2(\mathbb{R}^d; \mathbb{R}^d)$. The claim then follows as in the proof of Proposition 4.5.

Step 1: By the uniform bound on $\|\chi_{I_{N_k}} D_{N_k} u_{N_k}\|_{L^2(\mathbb{R}^d; \mathbb{R}^d)}$ there is a subsequence N_k and $v \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ such that

$$\chi_{I_{N_k}} D_{N_k} u_{N_k} \rightharpoonup v \text{ in } L^2(\mathbb{R}^d; \mathbb{R}^d).$$

Step 2: There is $u \in L^2_{\text{loc}}(\mathbb{R}^d)$ such that $v = Du$.

Proof: Fix $R > 0$ and let $\bar{u}_N := \frac{1}{|B_R|} \int_{B_R} u_N dx$. We use the discrete Poincaré inequality and Step 1 to see

$$\|u_N - \bar{u}_N\|_{L^2(B_R)} \leq C(R) \|D_N u_N\|_{L^2(B_R; \mathbb{R}^d)} \leq C(R).$$

Thus there is a subsequence N_k^R and $u_R \in L^2(B_R)$ such that

$$u_{N_k^R} - \bar{u}_{N_k^R} \rightharpoonup u_R \text{ in } L^2(B_R).$$

This can be done on arbitrary balls in \mathbb{R}^d , and by a diagonal sequence argument (consider the above subsequence N_k^R , on $B_{R'}$ it is also bounded, so there is a subsequence and a limit $u_{R'}$ on $B_{R'}$, but on $B_{R'} \cap B_R$ it must hold $u_R = u_{R'}$) there is $u \in L^2_{\text{loc}}(\mathbb{R}^d)$ such that on a subsequence

$$u_N - \bar{u}_N \rightharpoonup u \text{ in } L^2_{\text{loc}}(\mathbb{R}^d).$$

The rest of the argument is exactly as in the proof of Proposition 4.5.

Step 3: u satisfies the equation $\mathcal{A}u = \text{div}^* f$ and is thus unique up to the addition of constants.

Proof: As before (start with a function $\varphi \in C_c^1(B_R)$ and for $L^{(1-\alpha)N} > 2R$ extend φ to a function with period $L^{(1-\alpha)N}$ to deduce the weak form of the limit equation in \mathbb{R}^d with test function φ). \square

4.3 Smallness of error terms

Recall the definition of the large field regulator $w_N^{\Lambda_N}$ in (3.10).

Lemma 4.7

Let $f \in C^\infty(\mathbb{T}^d; \mathbb{R}^d)$. For $\xi = 0$ and $\xi = -\mathcal{C}^q \nabla_l^* f^N$ with $f^N = L^{-N\frac{d}{2}} f(L^{-N}x)$ there is a constant C , independent of N , such that

$$w_N^{\Lambda_N}(\xi) \leq C.$$

Proof. For $\xi = 0$ one computes $w_N^{\Lambda_N}(\xi) = 1$ (read carefully the definition of the large field regulator, see (3.10)). For $\xi = -\mathcal{C}^q \nabla_l^* f^N$ we use (4.4) in Lemma 4.3 for $\alpha = 1$ to see that

$$\nabla^s \mathcal{C}^q \nabla_l^* f^N(x) = L^{-N(\frac{d}{2}-1+s)} \mathcal{C}_N D_N^s D_{N,l}^* f(x') \quad \text{with } x = L^N x' \text{ and } x' \in \mathbb{T}^d.$$

Thus every growing factor L^N in $g_{N,x}(\xi)$ and $G_{N,x}(\xi)$ (see (3.9) and (3.8)) is perfectly annihilated. \square

Lemma 4.8

For $g^N(x) = L^{-\alpha N\frac{d}{2}} g(L^{-\alpha N}x)$, $g \in C_c^\infty(\mathbb{R}^d)$, it holds

$$|\mathcal{C}^q \nabla_l^* g^N|_{N, \Lambda_N} \leq C \tau(\alpha)^N$$

where C is independent of N and $\tau(\alpha) = L^{(1-\alpha)(\frac{d}{2}+4)}$.

Proof. We use Lemma 4.3 to get

$$\begin{aligned}
& |C^q \nabla_l^* g^N|_{N, \Lambda_N} \\
&= \max_{1 \leq s \leq 3} \sup_{x \in \Lambda_N} \frac{1}{h} L^{N(\frac{d-2}{2}+s)} |\nabla^s C^q \nabla_l^* g^N| \\
&\leq \max_{1 \leq s \leq 3} \sup_{x \in \Lambda_N} \frac{1}{h} L^{N(\frac{d-2}{2}+s)} \left| \sum_y C_N^q(x-y) (\nabla^*)^{s+1} g^N(y) \right| \\
&= \max_{1 \leq s \leq 3} \sup_{x \in \Lambda_N} \frac{1}{h} L^{N(\frac{d-2}{2}+s)} \left| \sum_y C_N^q(x-y) L^{-\alpha N(\frac{d}{2}+s+1)} (D_N^*)^{s+1} g\left(\frac{y}{L^{\alpha N}}\right) \right| \\
&= \max_{1 \leq s \leq 3} \frac{1}{h} L^{N(1-\alpha)(\frac{d-2}{2}+s)} \sup_{x' \in \Lambda'_N} \left| C_N (D_N^*)^{s+1} g(x') \right|.
\end{aligned}$$

Apply Corollary 4.2 to see

$$\sup_{x' \in \Lambda'_N} \left| C_N (D_N^*)^{s+1} f(x') \right| \leq C L^{2N(1-\alpha)} \|g\|_{C^{d+s+1}}.$$

Thus

$$|C^q \nabla_l^* g^N|_{N, \Lambda_N} \leq C \max_{1 \leq s \leq 3} \frac{1}{h} L^{N(1-\alpha)(\frac{d-2}{2}+s)} L^{2(1-\alpha)N} \leq C L^{N(1-\alpha)(\frac{d}{2}+4)}.$$

Set $\tau(\alpha) = L^{(1-\alpha)(\frac{d}{2}+4)}$ to obtain the claim. \square

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